

Introduction to Tree Logics

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(slides mostly based on the ones by W. Martens and T. Schwentick)

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Why Logic?

- Tree automaton \leftrightarrow algorithm
- Logical formula \leftrightarrow specification \leftrightarrow appropriate for queries
- Most expressive logics are tightly related to tree automata

Outline

1. Trees seen as logical structures
2. First-order logic over trees
3. Monadic second-order logic over trees

What is a Logic?

Mathematical logic

- A tool that allows reasoning (show *properties*) about *things* (e.g. arithmetic, set theory...)
- Before talking about a logic, one must define the *thing* we are talking about!
- The *thing* is usually a *structure* (unconstrained, word, tree...)

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Recall that in logic:

- A *relational vocabulary* is a sequence of relation names (R, \dots, T) with associated arities $\text{arity}(R), \dots, \text{arity}(T)$
- A *structure* S over (R, \dots, T) is a tuple

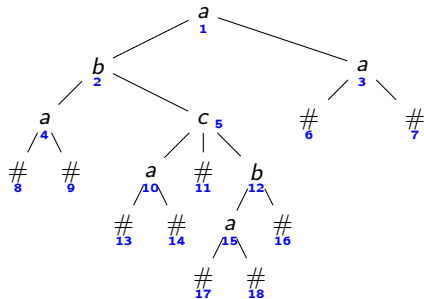
$$(D, R^S, \dots, T^S)$$

where D is a finite set, and $R^S \subseteq D^{\text{arity}(R)}, \dots, S^S \subseteq D^{\text{arity}(S)}$

Trees as relational structures

A tree t over a **ranked alphabet** $\Sigma = \{a, \dots, b\}$ naturally corresponds to a structure \underline{t} over the vocabulary $V_\Sigma = (\text{child}, <, L_a, \dots, L_b)$

Example



Structure

- $D^{\underline{t}} = \{1, 2, 3, \dots\}$
- L_σ : unary relations
 - $L_a = \{1, 3, 4, 10, 15\}$
 - $L_b = \{2, 12\}$
 - ...
- $\text{child}, <$ (ordering) are binary relations
 - $1 \text{ child}^{\underline{t}} 2$
 - $1 \text{ child}^{\underline{t}} 3$
 - $2 \text{ child}^{\underline{t}} 4$
 - ...
 - $2 <^{\underline{t}} 3$
 - $4 <^{\underline{t}} 5$
 - ...

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First-Order Logic over Trees

FO over the vocabulary $V_{\Sigma} = (\text{child}, <, L_a, \dots, L_b)$:

$$\begin{aligned} \varphi ::= & x = y \mid x \text{ child } y \mid x < y \mid L_a(x) \mid \dots \mid L_b(y) \\ & \mid \varphi \wedge \varphi \mid \neg \varphi \mid \exists x \varphi \end{aligned}$$

with usual abbreviations $\varphi \vee \varphi, \varphi \rightarrow \varphi, \forall x \varphi, \dots$

Example

All nodes labeled a in t have a child labeled b , iff:

$$\underline{t} \models \forall x (L_a(x) \rightarrow \exists y (L_b(y) \wedge x \text{ child } y))$$

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Notation/terminology

- If φ is a formula over V_Σ then

$$\text{Language}(\varphi) := \{t \mid t \text{ ranked tree } \Sigma \text{ such that } t \models \varphi\}$$

- φ is *satisfiable* iff $\text{Language}(\varphi) \neq \emptyset$, *valid* iff $\text{Language}(\neg \varphi) = \emptyset$
- We note $\text{FO}[\text{child}, <]$ as a shorthand for “FO over V_Σ ”
- A tree language L is definable in $\text{FO}[\text{child}, <]$ if there exists φ with $L = \text{Language}(\varphi)$

Applications to XML Trees

- A **closed** formula φ defines a **tree type** ($\text{Language}(\varphi)$), for example:

$$\forall x(L_a(x) \rightarrow \exists y(L_b(y) \wedge x \text{ child } y))$$

- A formula $\varphi(x, y)$ **with two free variables** defines a **relation between two tree nodes**: a (relative) query

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Exercise:

Is the following query definable in $\text{FO}[\text{child}, <]$?

- `child::a[parent::b]/child::c`
- `self::a/descendant::b`

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Example

- $\text{descendant}::p[\text{not child}::q]/\text{child}::*$
- $\exists z(x \text{ descendant } z \wedge L_p(z) \wedge \neg \exists w(z \text{ child } w \wedge L_q(w)) \wedge z \text{ child } y)$

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- Here, $z \text{ child } w$ stands for:
 $z \text{ descendant } w \wedge \neg \exists v(z \text{ descendant } v \wedge v \text{ descendant } y)$

Property

- Every XPath expression of the “CoreXPath” fragment is equivalent to a formula with 2 free variables in $\text{FO}^2[\text{descendant}, <]$
[Marx, 2004, Genevès and Vion-Dury, 2004]

CoreXPath

| | | | |
|------------------|-----|---|----------------------|
| <i>query</i> | ::= | <i>/path</i> | absolute path |
| | | <i>path</i> | relative path |
| | | <i>query</i> <i>query</i> | union |
| | | <i>query</i> \cap <i>query</i> | intersection |
| <i>path</i> | ::= | <i>path/path</i> | path composition |
| | | <i>path</i> [<i>qualifier</i>] | qualified path |
| | | <i>axis</i> :: <i>nodetest</i> | step |
| <i>qualifier</i> | ::= | <i>qualifier</i> and <i>qualifier</i> | conjunction |
| | | <i>qualifier</i> or <i>qualifier</i> | disjunction |
| | | not(<i>qualifier</i>) | negation |
| | | <i>path</i> | path |
| <i>nodetest</i> | ::= | σ | node label |
| | | * | any node label |
| <i>axis</i> | ::= | self child parent descendant preceding | tree navigation axis |
| | | descendant-or-self ancestor ancestor-or-self | |
| | | following-sibling preceding-sibling following | |
| | | | |

... what about the reciprocal property?

- Is any $FO^2[\text{descendant}, <]$ formula equivalent to a CoreXPath expression?
- A desirable property for a query language
- if true, CoreXPath is expressively complete with respect to paths definable in FO:
 - “if a path is definable in FO, then there exists a CoreXPath expression that defines it”.

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... No!

- CoreXPath is not complete with respect to $FO^2[\text{descendant}, <]$
- But a relatively simple extension of CoreXPath is sufficient for obtaining the reciprocal [Marx, 2004] ...

Example (CoreXPath extended)

$$e_1 \stackrel{\text{def}}{=} \text{self}::*\underbrace{(\text{child}::*[F_1])^*}_{e_2}/\text{child}::*[F_2]$$

- F_1 and F_2 are filtering expressions
- e_2 is not a CoreXPath expression, it defines the transitive-reflexive closure of the relation expressed by $\text{child}::*[F_1]$

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- F_1 and F_2 are filtering expressions
- e_2 is not a CoreXPath expression, it defines the transitive-reflexive closure of the relation expressed by $\text{child}::*[F_1]$
- e_2 is equivalent to:

$$x = y \vee (x \text{ descendant } y \wedge F_1(y) \wedge \forall z (x \text{ descendant } z \wedge z \text{ descendant } y \rightarrow F_1(z)))$$

- e_1 defines the set of paths (x, x) such that x has a descendant y for which F_2 is true and for all z strictly between x and y , F_1 is true.
- e_1 is not expressible in CoreXPath:
 - Any CoreXPath qualifier is equivalent to a FO formula with only 2 variables
 - It can be shown that the main qualifier of e_1 requires 3 variables

Example (CoreXPath extended)

$$e_1 \stackrel{\text{def}}{=} \text{self}::*[\underbrace{(\text{child}::*[F_1])^*}_{e_2}/\text{child}::*[F_2]]$$

- e_2 is called a **conditional axis** (different from descendant-or-self=child* since it depends on F_1)
- Conditional XPath is CoreXPath extended with all conditional axes in the 4 directions (descendant, ancestor, following-sibling, preceding-sibling)

Properties

- Conditional XPath is **expressively complete** with respect to FO[descendant, <] [Marx, 2004]
- Most CoreXPath properties (like linear-time evaluation) generalize to Conditional XPath

Is FO the Ideal Logic for XML?

Assessment for XML

- FO (and variants) are appropriate for characterizing the **expressive power** and (to some extent) the **complexity** of query languages...

Decidability and Complexity

- Over an unconstrained relational structure: FO is undecidable [Church, 1936, Turing, 1937]
- Over trees: FO is decidable but not elementary
- FO^2 (the fragment of FO restricted to only 2 distinct variables) : 2-NEXPTIME [Mortimer, 1975], NEXPTIME [Grädel et al., 1997]

But...

FO Expressivity is Limited

Limitation

- FO does not capture **all** regular tree languages
- Example:
 - The language L composed of all trees with an even number of a (recall tree automata!)
 - L is not definable in FO (exercise: intuitively, FO cannot “count”)
- FO is **not** expressive enough for reasoning over all DTDs, XML Schemas, Relax NGs

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Monadic Second-Order Logic

MSO is the extension of FO with set variables X :

$$\begin{aligned} \varphi ::= & x = y \mid x \text{ child } y \mid x < y \mid L_a(x) \mid \dots \mid L_b(y) \\ & \mid \varphi \wedge \varphi \mid \neg \varphi \mid \exists x \varphi \mid \mathbf{X}(x) \mid \mathbf{\exists X} \varphi \end{aligned}$$

with usual abbreviations $\varphi \vee \varphi, \varphi \rightarrow \varphi, \forall x \varphi, \forall X \varphi, \dots$

Example: Boolean Expressions

$$\begin{aligned} \text{true expression} \equiv & \exists X . X(\text{root}) \wedge \forall x \\ & (L_1(x) \rightarrow X(x)) \wedge \\ & (L_0(x) \rightarrow \neg X(x)) \wedge \\ & ((L_\wedge(x) \wedge X(x)) \rightarrow (\forall y[x \text{ child } y \rightarrow X(y)])) \wedge \\ & ((L_\vee(x) \wedge X(x)) \rightarrow (\exists y[x \text{ child } y \wedge X(y)])) \end{aligned}$$

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Notation and Terminology

- If φ is a closed formula over V_Σ then

$$\text{Language}(\varphi) := \{t \mid t \text{ ranked tree over } \Sigma \text{ such that } \underline{t} \models \varphi\}$$

- A tree language is definable in MSO if there exists a MSO formula φ with $S = \text{Language}(\varphi)$

MSO Variants: WSkS

- Recall the definition of our relational vocabulary
- We define the logical data model (ordered trees: **structure** + **order**)
- Instead of `child` and `<`, one can use the relations ch_i modeling the parent to i -th child relationship:

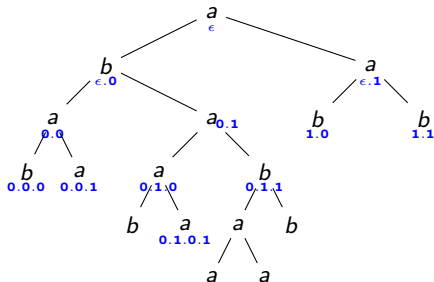
$x \text{ ch}_i y$ iff y is the i -th child of x

- Then, instead of $\text{MSO}[\text{child}, <]$, we can consider $\text{MSO}[ch_1, ch_2, \dots, ch_k]$
- Exercise: write some formulas
- This is **WSkS**: **Weak monadic Second-order logic of k Successors**
- WSkS is not so weak... ☺

WS2S: A Variant of Special Interest

- Zoom on $WSkS = MSO[ch_1, ch_2, \dots, ch_k]$ where $k = 2$
- Recall the encoding of unranked trees as binary trees
- Considering ch_1 and ch_2 is sufficient wlog!
- We can even call them first successor and second successor or “fc” (for first-child) and “ns” (next-sibling), explicitly referring to the unranked case
- Anyway, this is **WS2S**.

The logical tree structure: the WS2S perspective



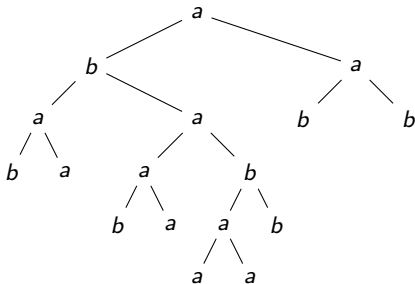
$\mathcal{L}_{WS2S} \ni \varphi ::= x = y \mid \text{firstchild}(x, y) \mid \text{nextsibling}(x, y) \mid L_a(x) \mid \dots \mid L_b(y)$
 $\mid \varphi \wedge \varphi \mid \neg \varphi \mid \exists x \varphi \mid X(x) \mid \exists X \varphi$

Logical Variants

- Now you know how to build and clearly define plenty of (different?) logical formalisms
- Most importantly: you understand why it is important to define and mention the **relational vocabulary** over the **logical structure**
- $FO[\text{child}, <]$, $FO[\text{descendant}, <]$, $FO[\text{fc}, \text{ns}]$, $FO[\text{fc}^+, \text{ns}^+]$, $FO^2[\text{fc}^+, \text{ns}^+]$
 - Exercise(s): what are the connections between those logics?
 - Btw: what was the problem again with $FO[\text{fc}, \text{ns}]$?
- $MSO[\text{child}, <]$, $MSO[\text{fc}, \text{ns}] = WS2S$
 - Exercise: finding MSO formulas not in any FO: e.g. true boolean expressions, other ones?

Exercise with MSO

Construct a WS2S formula φ which describes the set of trees with an even number of a nodes, e.g.:



$\mathcal{L}_{\text{WS2S}} \ni \varphi ::= x = y \mid \text{firstchild}(x, y) \mid \text{nextsibling}(x, y) \mid L_a(x) \mid \dots \mid L_b(y)$
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Expressivity and Decidability of MSO

Theorem [Thatcher and Wright, 1968, Doner, 1970]

- MSO exactly captures regular tree languages
- The satisfiability-problem for a MSO formula is decidable

Principle: the logic-tree automata connection

- Show that MSO is as expressive as tree automata
- We associate a tree automaton to each formula, and reciprocally
- For the direction logic \rightarrow automata, induction:
 - basic tree automata
 - $\varphi \vee \varphi'$: composition of tree automata (recall: closure of regular tree languages)
- Satisfiability of the logical formula: emptiness check of the final automaton

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\rightarrow We can compute the tree automaton (the algorithm) that corresponds to a given logical formula (specification)!

Complexity

What is the size of the tree automaton corresponding to φ ?

(a) smaller than φ

(b) same size as φ

(c) larger than φ

(d) HUGE

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Hints

- $\neg\varphi$: construction of the complement of the automaton for φ (determinisation)
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- Each quantifier alternation (e.g. $\exists\forall\exists$) introduces a determinisation...

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Theorem

- The number of states of the automaton associated with a formula with n quantifier alternations is in the worst case at least:

$$2^{2^{\dots 2^{c \cdot n}}}$$

where c is a constant. [Stockmeyer and Meyer, 1973]

- That's a lower bound!
- MSO satisfiability is decidable in non-elementary time (i.e. not bounded by any stack of exponentials)
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- ✗ You pay for what you get:
 1. Satisfiability is non-elementary
 2. Unless $P=NP$, there does not exist any elementary f (i.e. bounded by a stack of exponential) such that MSO formulas can be **evaluated** in time $f(\text{size}(\varphi)) \cdot p(\text{size}(t))$ with p polynomial [Frick and Grohe, 2002]
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 - ✗ This makes MSO almost useless as a query language :(
- There exists tree logics as **expressive as MSO** and for which both evaluation (model-checking) and satisfiability-checking can be implemented more efficiently (less complex): μ -calculus for trees [Genevès et al., 2015]



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